Approximate inference on planar graphs using Loop Calculus and Belief Propagation

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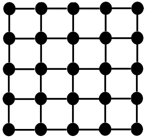
Outline

1. Motivation
2. Algorithm
3. Experiments
   - Setup
   - Full series
   - Grids

Motivation

- Exact inference on the Ising model defined on a **planar** graph is *easy* for **zero external fields** (Kasteleyn, Fisher and others, 1960s):



$$p(x) = \frac{1}{Z} e^{\sum_{(i,j)} w_{ij} x_i x_j}$$

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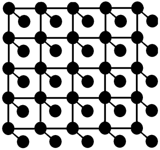

$$p(x) = \frac{1}{Z} e^{\sum_{(i,j)} w_{ij} x_i x_j + \sum_i \theta_i x_i}$$

Otherwise is **intractable**, $\#$P (Barahona, 82).

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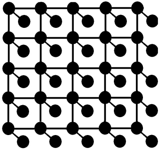

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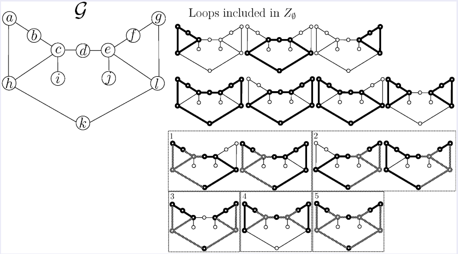
- Recently, the Fisher & Kasteleny method has been introduced in the Machine Learning community:
  - "Approximate inference using planar graph decomposition", Globerson A & Jaakkola T (NIPS 07)
  - "Efficient Exact Inference in Planar Ising Models", Schraudolph N & Kamenetsky D, (NIPS 08)
- Both perform **exact** inference on an *easy* planar model
- We directly **approximate** Z on *difficult* planar graphs.

Loop Calculus and Belief Propagation (BP)

- Exact Z of a general binary graphical model can be expressed as a **finite sum of terms** that can be evaluated once the BP solution is known. (Chertkov & Chernyak, 06a)

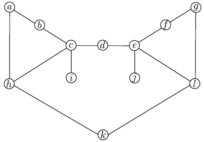
$$Z = Z^{BP} \cdot z, \quad z = \left(1 + \sum_{C \in \mathcal{C}} r_C\right)$$

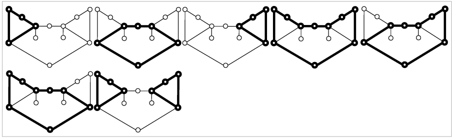


- Each term corresponds to a *generalized loop* (subgraph with no degree 1 vertices)

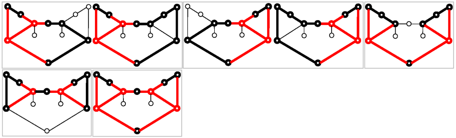
- Summing all terms is intractable... but truncation can provide improvements on BP (Gómez et al 07, Chertkov & Chernyak, 06b).

Motivation





2-regular loops



non 2-regular loops

2-regular loop : A loop where **all** nodes have degree two.

2-regular part. function $Z_\emptyset$: Approximation including all 2-regular loops only. $Z_\emptyset = Z^{BP} \cdot z_\emptyset$, $\quad z_\emptyset = 1 + \sum_{C \in \mathcal{C} s.t. |\bar{a}_C| = 2, \forall a \in C} r_C$.

Triplet : A node with degree 3 in the Forney graph.

Motivation

"Belief Propagation and Loop Series for planar graphs",
(Chertkov et al, 08)

- The 2-regular partition function $Z_\emptyset$ can be expressed as a sum of weighted perfect matchings.
- For **planar graphs**, $Z_\emptyset$ can be computed in **polynomial time**.
- The full loop series can be expressed as a sum over so-called **Pfaffian terms**, and each term may sum many loops.

Contribution

- We develop an algorithm to compute the full Pfaffian series.
- Empirical analysis:
  - Compare Loop and Pfaffian series
  - Analyze the accuracy of the $Z_\emptyset$ approximation.

Forney graph $\mathcal{G}$ with binary variables and nodes with degree at most 3.

1. Find stationary point of BP.
2. Obtain 2-core.
3. Construct planar embedding.
4. Obtain extended graph $\mathcal{G}_{ext}$.
5. Obtain Pfaffian orientation for the edges of the extended graph $\to \mathcal{G}'_{ext}$.
6. Construct skew-symmetric matrices $\hat{A}$ and $\hat{B}$.
7. The 2-regular partition function is:

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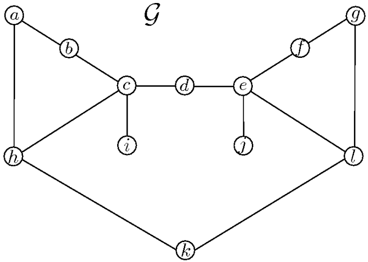

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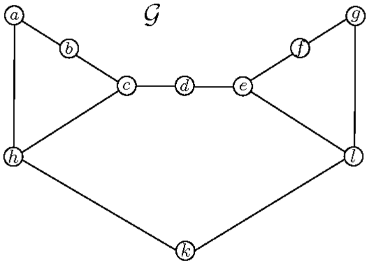

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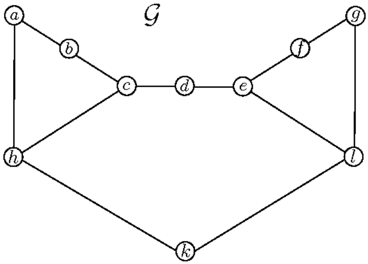

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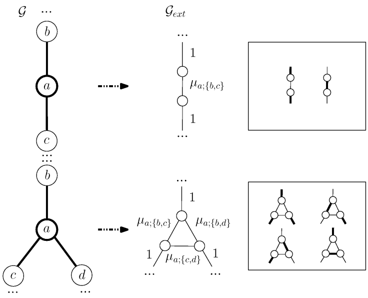

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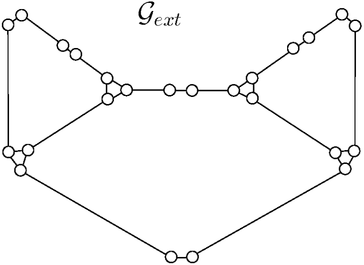

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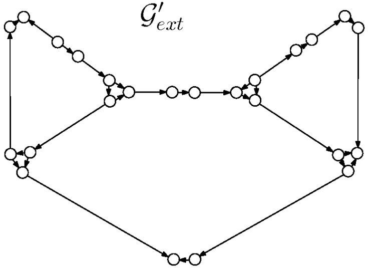

For every face, the number of clockwise oriented edges is odd.

Loop series for planar graphs

Computing 2-regular partition function

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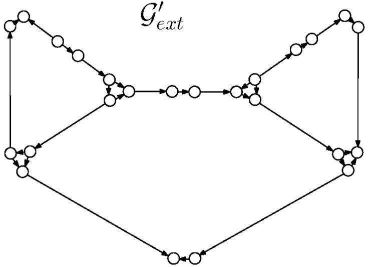

$$\hat{A}_{ij} = \begin{cases} +\mu_{ij} & \text{if } (i,j) \in \mathcal{E}_{\mathcal{G}'_{ext}} \\ -\mu_{ij} & \text{if } (j,i) \in \mathcal{E}_{\mathcal{G}'_{ext}} \\ 0 & \text{otherwise} \end{cases}.$$

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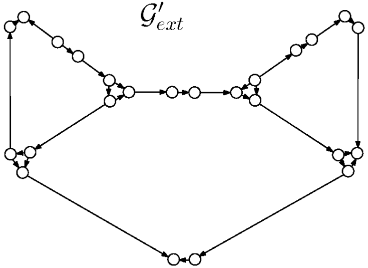

$$\hat{B}_{ij} = \begin{cases} +1 & \text{if } (i,j) \in \mathcal{E}_{\mathcal{G}'_{ext}} \\ -1 & \text{if } (j,i) \in \mathcal{E}_{\mathcal{G}'_{ext}} \\ 0 & \text{otherwise} \end{cases}.$$

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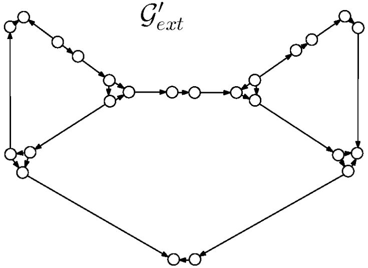

Computing full loop series

- Denote $\mathcal{T}$ the set of all possible triplets in $\mathcal{G}$.
- Consider a subset $\Psi \in \mathcal{T}$ with an even number of triplets.
- Loops in $\mathcal{G}$ including the triplets in Ψ correspond to perfect matchings on another extended graph $\mathcal{G}_{ext_\Psi}$.
- Exact Z can be written as a sum of Pfaffian terms:

$$z = \sum_{\Psi} Z_\Psi, \qquad Z_\Psi = z_\Psi \prod_{a \in \Psi} \mu_{a;\bar{a}}, \qquad z_\Psi = \text{sign}\left(\text{Pf}\left(\hat{B}_\Psi\right)\right) \cdot \text{Pf}\left(\hat{A}_\Psi\right).$$

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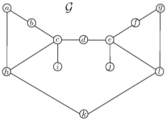

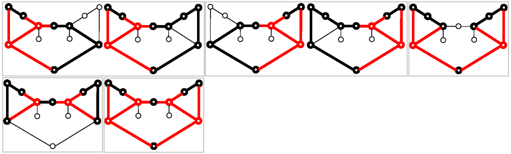

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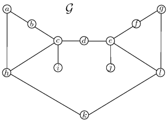

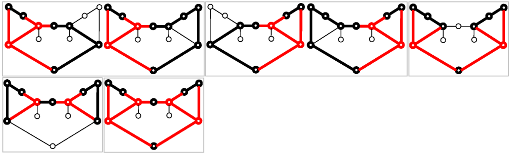

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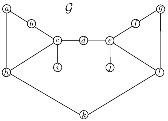

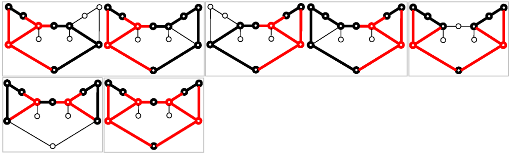

Loop series for planar graphs

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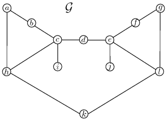
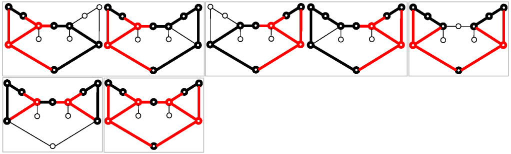


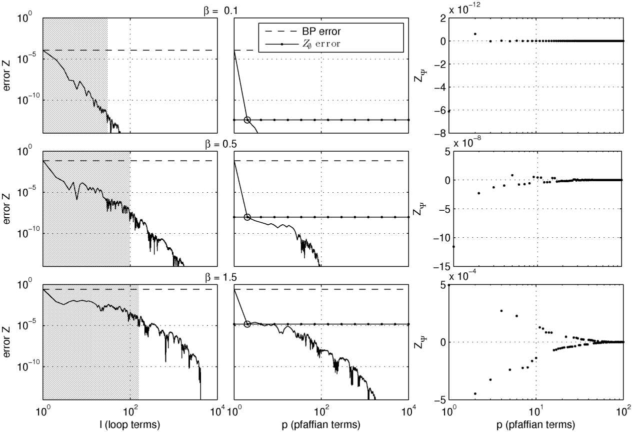


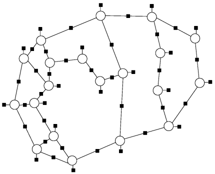
Model

- We consider binary pairwise models (Ising).
  - Interaction strengths $\{J_{a;\{ab,ac\}}\} \sim \mathcal{N}(0, \beta/2)$.
  - External fields $\{J_{a;\{ab\}}\} \sim \mathcal{N}(0, \beta\Theta)$.
- Θ and β determine how difficult the inference problem is.
- For $\Theta = 0$ problems are *easy*, i.e. $Z_\emptyset$ is exact.
- Error measure : error$Z' = \frac{|\log Z - \log Z'|}{\log Z}$.

Experiments

Full series





- A random instance: $\Theta = 0.1$ and $\beta \in \{0.1, 0.5, 1.5\}$.
- Both loop and pfaffian terms are sorted by absolute value in descending order.

We compare the $Z_\emptyset$ approximation with:

Truncated Loop-Series for BP (TLSBP): Gómez et al. (07').

Cluster Variation Method (CVM-Loopk): Heskes et al. (03').

Tree-Structured Expectation Propagation (TreeEP) : Minka & Qui (04').

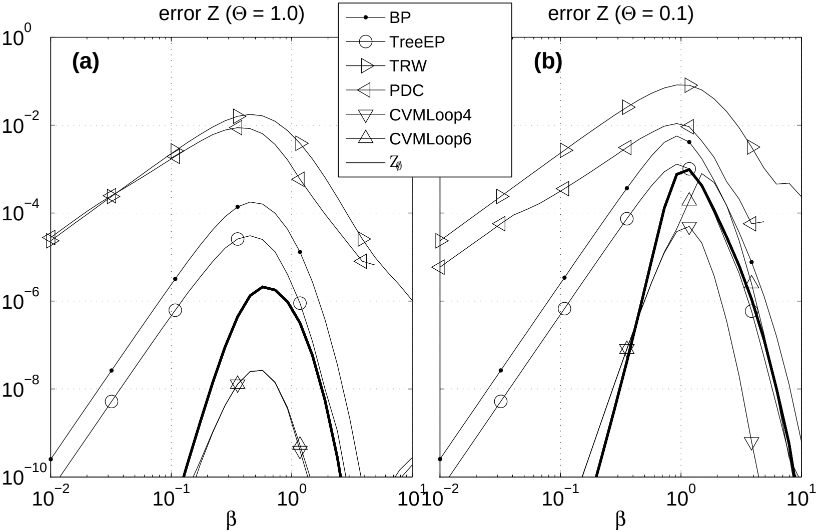
When possible, we also compare with the following two variational methods which provide upper **bounds** on the partition function:

Tree Reweighting (TRW) : Wainwright et al. (05').

Planar graph decomposition (PDC) : Globerson & Jaakola (07').

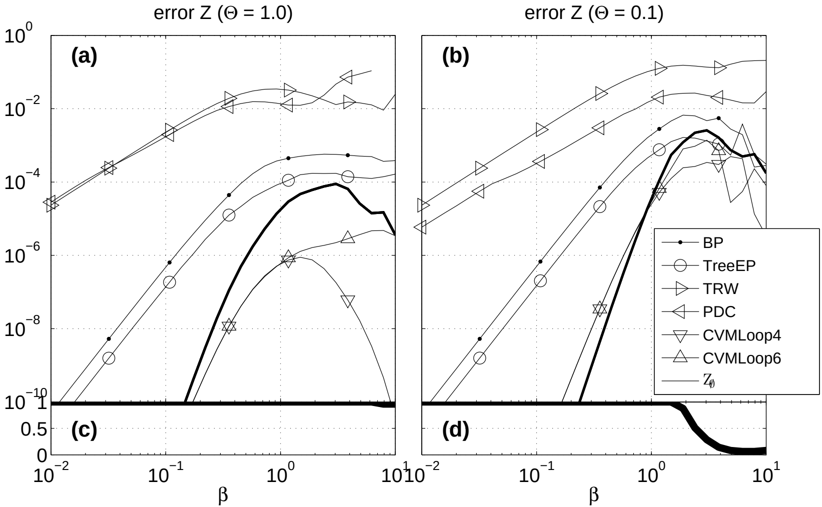
Experiments

Grids: Ising 7x7 (attractive interactions)



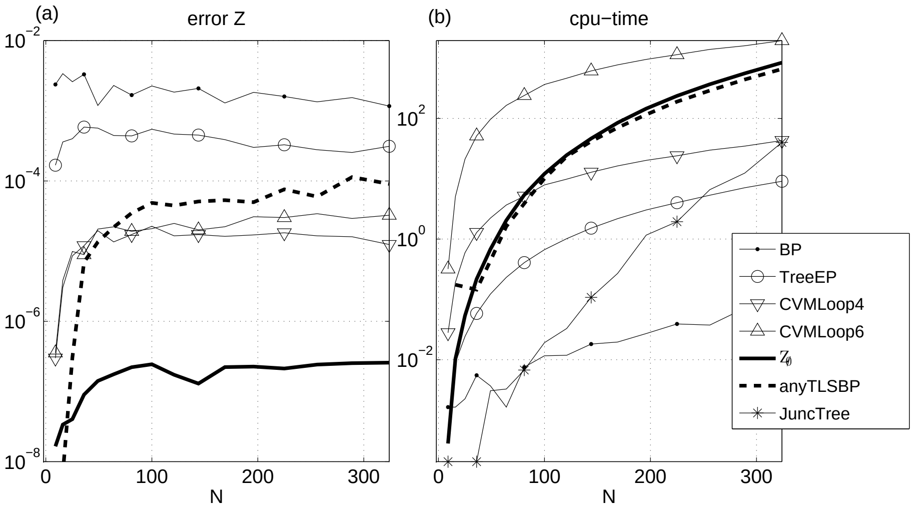
Experiments

Grids: Ising 7x7 (mixed interactions)



Experiments

Grids: scaling with model size (very weak local fields, $\Theta = 0.01$)



Conclusions

- Without the requirement of searching for loops, the $Z_\emptyset$ corrects the BP approximation even in difficult problems.
- Significant improvements are always obtained for sufficiently large external fields.
- $Z_\emptyset$ is competitive with other state of the art methods for approximate inference of Z.
- Computational cost: substitute brute-force evaluation of the Pfaffians by a smarter one available for planar graphs: $\mathcal{O}(N^3) \rightarrow \mathcal{O}(N^{3/2})$ (Gallucci 00', Loh and Carlso 06').
- Consider extensions to non-planar graphs.